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Oscillator death on small-world networks

Zhonghuai Hou and Houwen Xin*
Department of Chemical Physics, University of Science and Technology of China, Hefei, Anhui, 230026, People's Republic of China
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We have investigated the oscillator death behavior on small-world networks. On one hand, we find that small-world connectivity can eliminate the oscillator death present in the regular lattice. On the other hand, the small-world connectivity can also lead to global oscillator death which is absent in the regular lattice or the completely random network.

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I. INTRODUCTION

Recently, the study of complex networks has gained extensive attention [1–3]. An intriguing type of complex networks is the small-world network introduced by Watts and Strogatz [4–6]. The most striking feature of a small-world network is that it combines high clustering, which is usually found for regular lattices, and short characteristic path length, which is a typical property of random graphs. It has been shown that a lot of real systems are small-world networks, such as the cellular networks, metabolic networks, gene regulatory networks, etc. [1]. So far, the studies on small world can be divided into two main categories. The dominant one is to study the topological properties of small-world networks and various mechanisms to determine the topology. The other one, which is more important, is to study how the small-world topology can influence the system's dynamic features [2]. Recently, it was found that any spreading rate can lead to the whole infection of disease in a "scale-free" small-world network [7], stochastic resonance [8] and synchronization [9] can be considerably improved on small-world networks, and small-world connections can greatly enhance the probability of spiral wave formation in excitable media [10]. It shows that small-world connectivity plays a crucial role for the system's dynamics.

In the present Rapid Communication, we study the collective dynamic behaviors of an array of coupled limit cycle oscillators on small-world networks. Coupled limit cycle oscillators provide a simple but powerful mathematical model for simulating the collective behavior of a wide variety of systems that are of interest in physics, chemical, and biological sciences [11]. In general, there could be a wide variety of collective behaviors, such as phase synchronization, phase trapping, Hopf bifurcation, and even chaos. Here we focus on the amplitude death behavior which may occur when the coupling is strong enough and there is a constant frequency gradient along the chain. Amplitude death is of interest because it may apply to the arrhythmia of the network of cardiac pacemakers, [12] and thermo-optical oscillators [13], as well as other examples such as chemical reactions taking place in coupled stirred tank reactors [14]. It is interesting to investigate how the small-world topology would affect the oscillator death behavior. In the present work, we find that small-world connectivity can effectively eliminate the oscillator death present in the regular chain. On the other hand, small-world topology can also lead to global oscillator death, which is absent in the regular lattice or a completely random network.

II. MODEL AND RESULTS

The small-world networks are constructed based on the Watts and Strogatz (WS) model [5], which is obtained by randomly *rewiring* some edges of a regular chain. We start from a one-dimensional regular lattice with free boundary condition, composed of $N=102$ elements with each vortex connected to its $K=4$ nearest neighbors. The number of rewiring edges is given by Mp, here $M=\Sigma_{i=1}^{K/2}(N-i)$ is the total number of edges in the network, and p is the rewiring probability. For $0<p<1$, we obtain a disordered network which lies between a regular lattice ($p=0$) and a completely random network ($p=1$). One may use the parameter p to measure the randomness of the network, but we should note that for a given p, there could be a lot of network realizations.

We use the equations for slow complex amplitudes $\dot{z}_j=|z_j|\exp(i\varphi_j)$ in the form [15]

$$\dot{z}_j=i\omega_j z_j+(r-|z_j|^2)z_j+d\sum_i (z_i-z_j). \tag{1}$$

Here $\omega_j=\Delta\omega(j-1)/(N-1)$ is the frequency of the jth element ($\Delta\omega=\omega_N-\omega_1$ is the frequency range along the chain), d is the coupling constant, $r=0.5$ is the oscillation growth rate, and the summation goes over all the vortices coupling to site j. For a regular lattice, regions of amplitude death are formed when $\Delta\omega$ and d are large enough, even if the conditions of self-oscillation are fulfilled for each vortex in the absence of coupling. Figure 1 shows such an example for $d=2.0$, and $\Delta\omega=5.0$. One should note that the oscillation amplitude $|z_j|$ inside the death region is not exactly zero but very small. Since the death region only exists in the middle part of the chain, we may call it *partial* death to compare with the *global* death when such region extends to all the vortices.

To qualitatively manifest the intensity of the collective oscillation of the network, we choose the normalized mean "incoherent" energy E and "coherent" energy W as the characteristic functions, which are defined as [15]

*Corresponding author. Email address: hzhlj@ustc.edu.cn

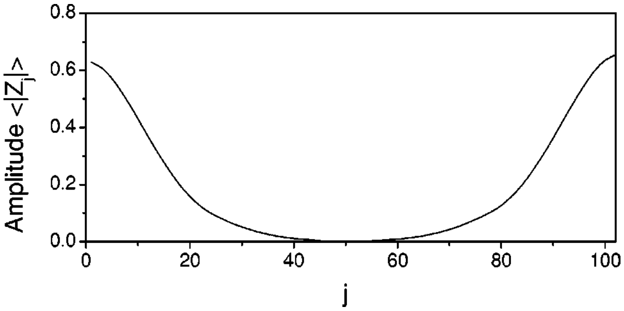

FIG. 1. Oscillator death in the regular chain ($p=0$) with constant frequency gradient. The frequency range is $\Delta\omega=5.0$, coupling constant is $d=2.0$; $N=102$, $K=4$. Plotted is the time-averaged oscillation amplitude.

$$E=\left[\frac{\langle \Sigma_{j=1}^{N}|z_j|^2\rangle}{Nr}\right] \quad \text{and} \quad W=\left[\frac{\langle |\Sigma_{j=1}^{N} z_j|^2\rangle}{N^2 r}\right].$$

Here $\langle\cdot\rangle$ denotes averaging over time and $[\cdot]$ stands for averaging over 40 different network realizations for each p. A large E means relatively large average oscillation amplitudes, while a larger W implicates more synchronous oscillations. When amplitude death is present, both E and W have relatively small values.

The dependence of E and W on p are depicted in Figs. 2(a) and (b) for $d=2.0$ and $\Delta\omega=5.0$. The curves can be divided into four stages. In stage 1 and 2, E and W first increase and then decrease to nearly zero, showing a peak around $p\sim0.02$. E and W remain nearly zero in stage 3 and then increase again in stage 4. Therefore, we find that the network's topology has multiple effects as described below.

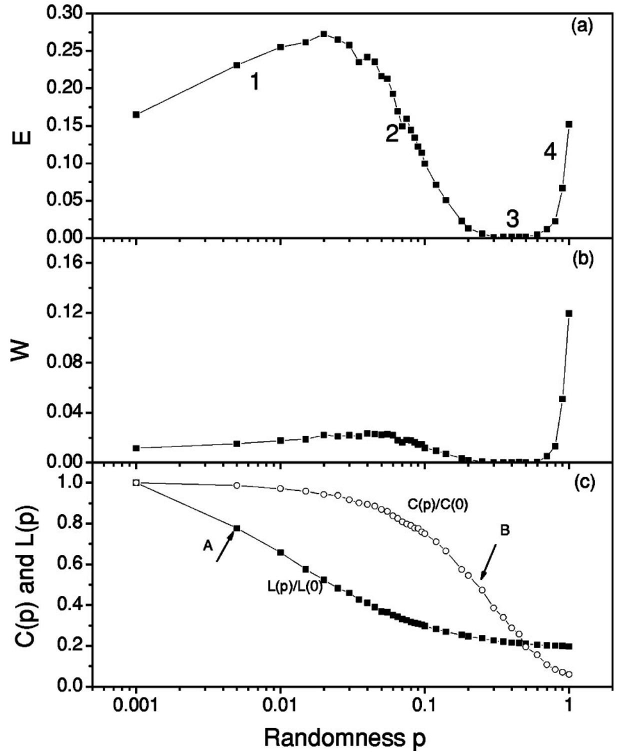

FIG. 2. (a) Dependence of E on p; four stages are present as indicated by 1, 2, 3, and 4. (b) Dependence of W on p. (c) The dependence of clustering coefficient $C(p)$ and characteristic length $L(p)$ on the randomness p. The onset and end of the SW region is approximately labeled by points A and B. Note that the data for $p=0.001$ is actually the same as $p=0$ due to the finite system size N and this is the case for all later figures in the present paper.

(i) *Death elimination: A small fraction of shortcuts can eliminate the partial amplitude death*. This is demonstrated by stage 1. In addition, there exists an optimal level of topological randomness when the incoherent and coherent energy of the system reaches a maximum. Near the peak $p\sim0.02$, the number of random shortcuts is only about 4, which is much smaller than the total number of edges, but the network already has typical small-world (SW) features, characterized by a large clustering coefficient $C(p)$ and a small characteristic path length $L(p)$. Here $L(p)$ and $C(p)$ are defined in a standard way as in Ref. [5], $L(p)$ measures the typical separation between two vertices (a global property), and $C(p)$ measures the cliquishness of a typical neighborhood (a local property). In Fig. 2(c), the dependences of $L(p)$ and $C(p)$ on p are shown as a reference. Though the boundary of the SW region has not been uniformly defined yet, it is generally accepted that the onset of the SW region scale as $1/(NK/2)$,[16] while the end of it may be defined as the point when $C(p)/C(0)$ reduces to 0.5. We can see that the peak positions in Figs. 2(a) and (b) lie exactly inside the SW region.

(ii) *Global death: Intermediate randomness may lead to global amplitude death*. This is demonstrated by stage 3 where both E and W are nearly zero. The onset of this stage is approximately the end of the SW region. It is interesting to note that inside this region, the global death is robust to different realizations of the small world. Note for each p, we have performed the numerical experiments on 40 different network realizations. Though each realization has different network topology, they all lead to global death. This robustness implies that the system's dynamic behavior could depend only on some collective property of the network, such as the average randomness p, rather than the detailed topological structure.

(iii) *Synchronization: Large randomness can help oscillator synchronization*. This is demonstrated by stage 4, where both E and W increase. The sharp increment and relatively large value of W indicates that the network vortices tend to be synchronized when enough long-range connections exist in the network. This is in consistent with the results of recent works that SW connectivity can help synchronization of coupled nonlinear oscillators.[9]

Accordingly, the spatial profiles of the average oscillation amplitude for typical p are shown in Fig. 3. For each vortex j, the amplitude $\langle|z_j|\rangle$ is averaged over time and different network realizations. It is clear that the death region in the regular chain ($p=0$) is effectively eliminated for $p=0.02$. The whole chain drops to global death for $p=0.3$, and reaches a nearly homogeneous oscillating state for $p=0.7$ and 1.0.

In Fig. 4, the dependences of these effects on the fre-

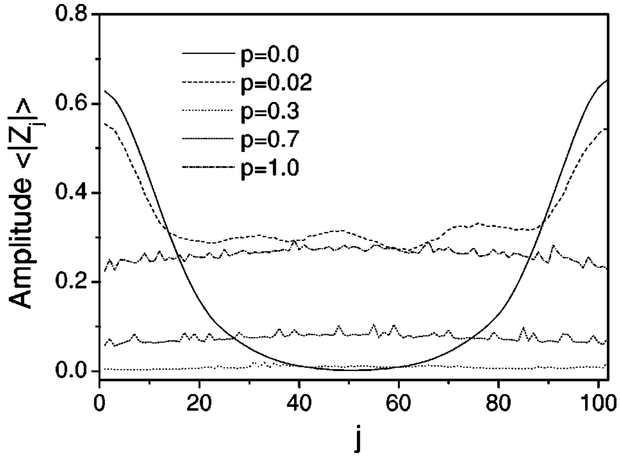

FIG. 3. Amplitude profile for different p. For $p=0.02$, the death region in the regular chain is obviously eliminated. For $p=0.3$, global amplitude death is obtained. For $p=0.7$ and 1.0, nearly homogeneous oscillation profiles are observed, which implies that the network is nearly synchronized.

quency scatter $\Delta\omega$ and coupling constant d are shown. When $\Delta\omega$ increases for a fixed d [Fig. 4(a)], the ''death-elimination'' and ''synchronization'' effects are reduced, while the ''global-death'' effect is enhanced. If we keep $\Delta\omega=5.0$ and change d [Fig. 4(b)], we find that a larger d tends to reduce the death-elimination effect and enhance the synchronization effect; the width of stage 3, where the global-death effect dominates, remains nearly unchanged, but the onset of it moves to smaller p.

To further understand the role of topological randomness, we have also performed similar studies on an alternative type

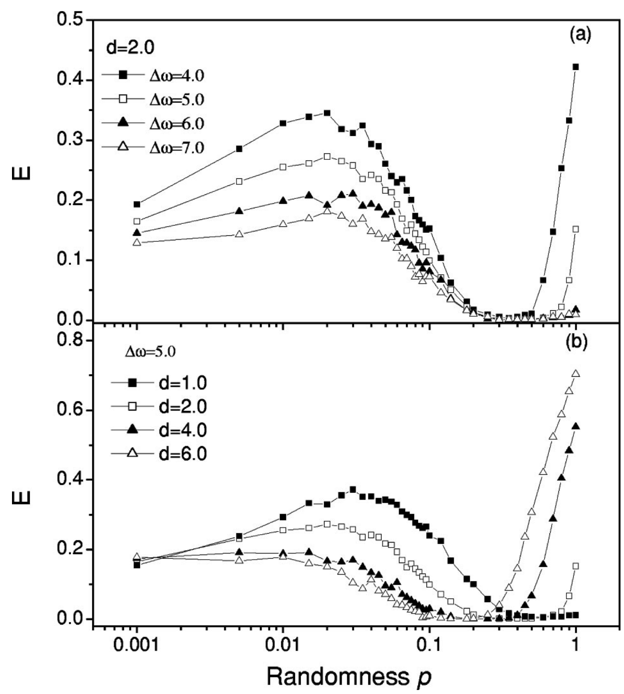

FIG. 4. Dependences of E on p for (a) different $\Delta\omega$ and (b) different d.

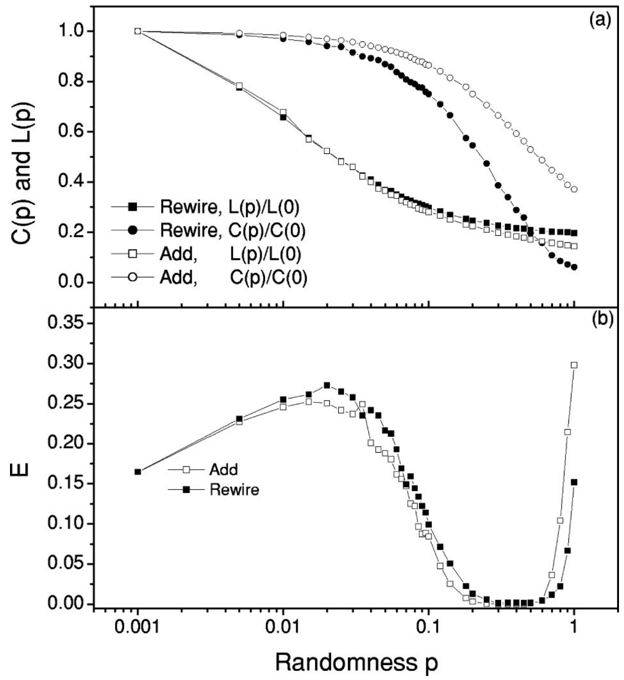

FIG. 5. (a) Comparison of the network properties of the WS model (Rewire) and the growing model (Add). The growing model shows a slightly larger $C(p)$ and smaller $L(p)$. (b) Comparison of the incoherent energy E. See text for discussions.

of small-world network, which is obtained by randomly *adding* (not rewiring) shortcuts to the original regular chain [16]. To keep correspondence with the WS model, we can also quantify the randomness of the network by p, which is the ratio of the number of added shortcuts to the total edges M in the original regular chain. As the WS model, this ''growing'' small-world also has similar dependences of characteristic path length and clustering coefficient on p as depicted in Fig. 5(a), whereas the growing network has a slightly larger $C(p)$ and smaller $L(p)$ for the same p. To see what this difference would affect the system's dynamics, the dependence of E on p for $d=2.0$, $\Delta\omega=5.0$ for both cases are shown in Fig. 5(b). The situation of W is similar and not shown here. The spatial profiles are quite similar to Fig. 3, but the amplitude for large p is much larger for the growing network. It seems that around the region where E reaches the maximum ($0.01<p<0.1$), larger clustering tends to hinder the death-elimination effect, which leads to a smaller E; whereas when p is large and the synchronization effect dominates, smaller characteristic path length will make synchronization easier, leading to a larger E. However, more detailed numerical studies and possibly theoretical analysis are required to get a deep insight into such phenomena.

It is now well known that spatial disorder may play constructive roles in spatial-extended systems. For example, disorder can eliminate oscillator death in coupled oscillators [15], tame spatiotemporal chaos in coupled pendulums [17], sustain spiral waves in excitable media [18], etc. Our findings here show that topological disorder can also play constructive roles. Recently, we have also found that a small fraction of shortcuts can effectively tame the spatiotemporal chaos observed in an array of coupled chaotic oscillators [19]. Such constructive effects of topological disorder in complex systems may deserve more and more attention in future works.

III. CONCLUSION

To conclude, we have studied the collective dynamics of a chain of couple oscillators on small-world networks. It is found that the small-world connectivity plays nontrivial roles on the oscillator death behavior. On one hand, a small fraction of random shortcuts can significantly eliminate the oscillator death observed in the regular chain. On the other hand, small-world connectivity can also lead to global amplitude death which is not present in the regular random network. Since many real systems (cellular networks, gene regulatory networks, protein networks, etc.) may have small-world features, and their collective dynamics could be modeled by coupled oscillators, these results may find a variety of applications. Our study may also stimulate further investigation on the role of network topology on system's dynamics.

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